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Sinc Collocation-Interpolation Method for the Simulation of Nonlinear Waves

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Abstract—Numerical simulation is of great importance in the study of nonlinear waves, where several delicate aspects must be dealt with. The presence of spatial and time gradients, of complex nonlinearities and high order derivatives, in fact generates various complex problems. This paper develops a collocation-interpolation method based on interpolation by the Sinc functions. After a description of the method, several examples are shown. These show the good results that can be obtained, the efficency and stability of the method, and the low computing time. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Collocation-interpolation methods, Sinc functions, Solitons, Korteweg-de Vries, Sine-Gordon, Nonlinear waves,.

1. INTRODUCTION

The study of nonlinear wave phenomena is one of the relevant topics in fluid mechanics, and even though this field of research has a very long both theoretical and experimental tradition (e.g., [1,2]), new ideas and interesting aspects are continously emerging. The comprehension of the role of the nonlinearities, influence of boundary and initial conditions, existence of chaotic behaviours, investigation of spatial or temporal patterns, and the study of their stability are outstanding and challenging research topics [3–7]. The mathematical complexity of the partial differential equations that describe these waves makes numerical calculation have a fundamental role in research, and it is this complexity that has led to the setting up of numerical methods which are suitable for a correct simulation of waves. The presence of strong gradients in space and in time and the existence of nonlinear interactions require the use of stable and robust algorithms [8–10].

This paper is devoted to show that for the simulation of nonlinear waves, reliable results can be obtained by generalized collocation interpolation methods based on the use of Sinc functions [11]. It is well known that a commonly applied solution technique of nonlinear initial-boundary value problems for partial differential equations is the differential quadrature method by Bellman and Casti [12], reviewed in [13]. This method was developed by several authors in the deterministic [14] and stochastic framework [15], as it was documented in the review paper [16] which

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provides a detailed report on the application and developments of the method revisited under the denomination generalized collocation method.

This method discretizes the original continuous model (and problem) into a discrete (in space) model, with a finite number of degrees of freedom, while the initial-boundary value problem is transformed into an initial-value problem for ordinary differential equations. A relevant feature of the above-mentioned developments consists of replacing the classical Lagrange interpolation, which may not be useful to deal with problems in unbounded domains and with solutions that are oscillating with high frequency in the space variables, by a suitable use of Sinc functions, which are characterized by spectral approximation properties. This method will be summarized in the sections which follow with reference to the above-cited review paper [16]. Further analysis can be recovered in [17], where the treatment of nonlinear boundary conditions is developed and the software Mathematica is applied to optimize the treatment of stiff systems of ordinary differential equations.

The analysis of nonlinear waves proposed in this paper needs a proper development of methods to deal with multiple boundary conditions (Dirichlet and Neumann) related to the statement of evolution problems for equations with high-order space derivatives, although we cannot naively claim that other methods for the treatment of hydrodynamical and transport equations [18–20] do not efficiently work. Nevertheless, the treatment of some interesting test cases shows that the present method, as we shall see, provides careful solutions to various initial value problems.

After a description of the method in the next section, some exemplary cases are subsequently discussed. These concern three classical nonlinear wave equations: the generalized third- and fifth-order Korteweg-de Vries equations and the sine-Gordon equation. The importance of these models, the complexity of the waves that they describe, and the possibility of comparison with other methods have dictated this choice. Moreover, when possible, comparisons with analytical results are shown. Finally, some comments on the computation efficiency and numerical stability of the method are proposed.

2. NUMERICAL METHOD

Consider a nonlinear partial differential waves equation of the type

$$\frac{\partial u}{\partial t} = f\left(t, x, \frac{\partial u}{\partial x}, \dots, \frac{\partial^r u}{\partial x^r}\right), \tag{2.1}$$

where t is the time, x is the space, and u is the dependent variable

$$u = u(t,x) : [0,T] \times [a,b] \to [u_{\min}, u_{\max}], \tag{2.2}$$

in which T, a, and b define the time-spatial domain of interest and $u_{\min}$ and $u_{\max}$ are the limit values of u.

In the problems concerning waves simulation, it is very often supposed that the dependent variable and all its spatial derivatives decay to zero at the boundaries $x = a$ and $x = b$ of the spatial domain. Specifically, we deal with problems with two Dirichlet boundary conditions

$$u(t,a) = 0, \qquad u(t,b) = 0, \tag{2.3}$$

and $(r-2)$ Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,a) = 0, \qquad \frac{\partial u}{\partial x}(t,b) = 0, \qquad \dots. \tag{2.4}$$

Moreover, the initial condition

$$u(0,x) = u_{t=0}(x), \qquad \forall\, x \in [a,b], \tag{2.5}$$

is given.

The numerical simulation of the above initial-boundary value problem can be developed through the following three steps.

STEP 1. The space variable is discretized into the collocation

$$i = 1, \ldots, n : \{x_1 = a, \ldots, x_i = a + (i-1)h, \ldots, x_n = b\}, \qquad h = \frac{|b-a|}{n-1}, \tag{2.6}$$

so that the independent variable $u = u(t,x)$ is interpolated and approximated by means of the Sinc function as follows:

$$u(t,x) \cong u^n(t,x) \equiv \sum_{j=1}^{n} S_j(x;h) u_j(t), \tag{2.7}$$

where $u_j(t) = u(t, x_j)$ denotes the values of u in the nodal points of the chosen collocation and

$$S_j(x;h) = \frac{\sin z_j}{z_j}, \qquad z_j = \frac{\pi}{h}(x - (j-1)h - a), \qquad S_j(x_i) = \delta_{ij}, \tag{2.8}$$

where δ_{ij} is the Kronecker delta function. In this way, the spatial behavior and the time evolution of the solution are decoupled: the Sinc functions describe the spatial behavior of the solution based on the time evolution of the dependent variable in the nodal points.

STEP 2. According to the collocation-interpolation defined in the previous step, the partial derivatives of the variable u in the nodal point of collocation are approximated as follows:

$$\frac{\partial^r u}{\partial x^r}(t;x_i) \cong \frac{\partial^r u^n}{\partial x^r}(t;x_i) = \sum_{j=1}^{n} a_{ji}^{(r)} u_j(t), \tag{2.9}$$

where

$$a_{ji}^{(r)} = \frac{d^r S_j}{dx^r}(x_i). \tag{2.10}$$

Technical calculations provide the relationships for the derivative coefficients that can be computed by the following recurrence formulas:

$$a_{ji}^{(2r)} = \frac{(-1)^{i-j}}{h^{2r}(i-j)^{2r}} \sum_{k=0}^{r-1} (-1)^{k+1} \frac{2r!}{(2k+1)!} \pi^{2k} (i-j)^{2k}, \qquad a_{ii}^{(2r)} = \left(\frac{\pi}{h}\right)^{2r} \frac{(-1)^r}{(2r+1)}, \tag{2.11}$$

for even coefficients, where $r = 1, 2, \ldots,$ and

$$a_{ji}^{(2r+1)} = \frac{(-1)^{i-j}}{h^{2r+1}(i-j)^{2r+1}} \sum_{k=0}^{r} (-1)^k \frac{(2r+1)!}{(2k+1)!} \pi^{2k} (i-j)^{2k}, \qquad a_{ii}^{(2r+1)} = 0, \tag{2.12}$$

for odd ones.

STEP 3. Substituting expressions (2.11),(2.12) of the partial derivatives in equation (2.1) yields a system of ordinary differential equations which describes the evolution of the values $u_i(t)$ of the variable in the nodal points, that is,

$$\frac{du_i}{dt} = f(t, x_i, \mathbf{u}, \mathbf{a}), \tag{2.13}$$

for $i = 1, \ldots, n$ and where $\mathbf{u}(t) = \{u_i(t)\}$ and $\mathbf{a} = \{a_{ji}, a_{ji}^{(1)}, \ldots\}$.

The Dirichlet boundary conditions (2.3) are then enforced by putting

$$u_1 = 0, \qquad u_n = 0. \tag{2.14}$$

To show how the Neumann boundary conditions can be introduced, the case $\frac{\partial u}{\partial x}(t,a) = 0$ is used. By considering interpolation (2.7), this condition can be rewritten according to equation (2.9), that is,

$$\frac{\partial u}{\partial x}(t, x_1) = \sum_{j=1}^{n} a_{j1}^{(1)} u_j(t) = 0, \tag{2.15}$$

which, with (2.14), yields

$$a_{21}^{(1)} u_2(t) = -\sum_{j=3}^{n-1} a_{j1}^{(1)} u_j(t), \tag{2.16}$$

and so

$$\frac{du_2}{dt} = -\frac{1}{a_{21}^{(1)}} \sum_{j=3}^{n-1} a_{j1}^{(1)} \frac{du_j}{dt}. \tag{2.17}$$

The same passages can be followed for boundary conditions involving higher spatial derivatives, considering the correspondent coefficients $a_{ji}^{(r)}$.

System (2.13) coupled to the conditions (2.14) and to those of type (2.17) has to be linked to the initial condition

$$u_i(0) = u_{t=0}(x_i), \tag{2.18}$$

for $i = 2, \dots, n-1$. The solution of the initial-boundary value problem is then obtained solving the initial value problem (2.13)–(2.18), by suitable standard methods for ordinary differential equations (see [21,22]), and interpolating the solution by the method used in Step 2.

Finally, some considerations are necessary referring to (2.7). The convergence of the series u^n to u with a decrease in step h, that is, with an increase of n, is assured under two conditions [13]:

(1) the function u must belong to the functional Paley-Wiener space, and
(2) u must be defined on the whole real line; that is, a and b must tend to infinity.

The first condition is practically always verified in the simulation of waves phenomena. On the contrary, the possibility to simulate only a finite number of nodes prevents the second condition from being satisfied. One possible solution is to introduce a change of variable (e.g., by means of a logarithmic transformation of the spatial variable x) and to project the limited spatial domain $[a, b]$ into the whole real axes $]-\infty, \infty[$. However, it is necessary to note that, in numerical investigation of the dynamics of waves, this approach is very often not necessary. As already mentioned, it is almost always assumed that the dependent variable u decays to zero to the extrema of the spatial domain. In this case, the summary presented in (2.7) coincides with that involving the whole real axes. Thus, the second condition for the convergence results to be practically of no influence on the application dealt with here.

3. TEST CASES

The generalized collocation-interpolation method based on the use of the Sinc function allows us to simulate waves equations with different types of nonlinearities. In this paper, some classical models are investigated to demonstrate the capability of the numerical scheme. A soliton solution for the generalized third-order Korteweg-de Vries (hereafter KdV) equation, fifth-order KdV equation, variable-coefficient KdV equation, and sine-Gordon equation are considered and in all cases the dependent variable is assumed to decay to zero at the simulation domain boundary. These equations are chosen primary because they involve different types of nonlinearities and numerical difficulties; moreover, as in some cases the analytical solution of the problem is known, useful comparisons with numerical results are possible. Finally, these models are often used as test cases to check numerical methods, and therefore, become reference cases in the numerical community.

EXAMPLE 1. GENERALIZED THIRD-ORDER KORTEWEG-DE VRIES EQUATION. The first example regards the generalized third-order Korteweg-de Vries equation. This arises in several fields of fluid mechanics—such as water waves, internal gravity waves in a stratified fluid, or waves in a rotating atmosphere (Rossby inertial waves) [1,23], and one of its possible forms is [23]

$$u_t + u^m u_x + \mu u_{xxx} = 0, \tag{3.1}$$

where the nonlinear term $u^m u_x$ balances the dispersion one μu_{xxx}.

EXAMPLE 1.1. SINGLE SOLITON. One of the analytical solutions of (3.1) is the solitary wave

$$u(t,x) = \left[A \operatorname{sech}^2(\kappa x - \omega t - x_0)\right]^{1/m}, \tag{3.2}$$

where $A = 2(m+1)(m+2)m^{-2}\mu\kappa^2$ and $\omega = 4\mu\kappa^3 m^{-2}$. As a first example, we choose $m = 1$ and the same values used by Djidjeli *et al.* [10]: $\mu = 1$, $\kappa = 0.3$, and $x_0 = 0$. The initial condition is given by (3.2), with $t = 0$, while the space and time intervals of simulation are $[-30, 30]$ and $[0, 40]$, respectively.

Figure 1 shows the results obtained with $n = 60$ and $\Delta t = 0.01$ (note that in this method it is not necessary to choose a number of 2^k nodes as in the methods based on FFT (e.g., [10]). The results are in good agreement with the analytical solution: the L_∞ norm of the absolute error at the end of the simulation ($t = 40$) is $8.98 \cdot 10^{-4}$. The CPU time for a Pentium III 550 MHz is 3.3 s.

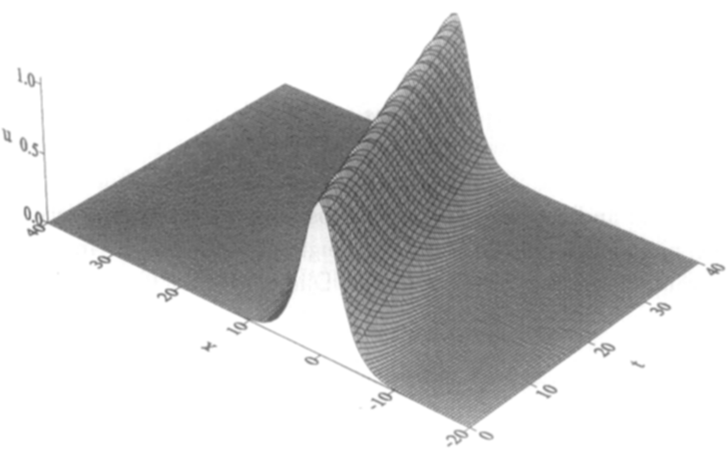

Figure 1. Generalized third-order KdV equation with $m = 1$: one soliton.

Another way of proving the good agreement of the result is to calculate the invariant quantities given by the conservation laws of mass, momentum, energy, etc, ... Without entering into detailed description we here recall that, for the generalized third-order KdV equation [1], it is possible to find an infinite number of conserved densities that, integrated over the whole range of the space domain, are independent of time [23,24]. In the case where $m = 1$, the first five invariant quantities are given by

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} u\,dx, \\ I_2 &= \int_{-\infty}^{+\infty} \frac{1}{2}u^2\,dx, \\ I_3 &= \int_{-\infty}^{+\infty} \left(u^3 + \frac{1}{2}u_x^2\right) dx, \end{aligned} \tag{3.3}$$

$$I_4 = \int_{-\infty}^{+\infty} \left(5u^4 + 10uu_x^2 + u_{xx}^2\right) dx,$$
$$I_5 = \int_{-\infty}^{+\infty} \left(21u^5 + 105u^2u_x^2 + 21uu_{xx}^2 + u_{xxx}^2\right) dx. \tag{3.3(cont.)}$$

In particular, the conserved densities I_1, I_2, and I_3 refer to the mass, momentum, and energy conservation, respectively. Substituting (3.2) in (3.3) yields $I_1 = 7.200$, $I_2 = 2.592$, $I_3 = 4.666$, $I_4 = 23.131$, and $I_5 = 102.792$. Table 1 shows the value of L_∞ of the invariant quantities compared with the analytical values, at the start, middle, and end values of the time interval of simulation. Note that the I_4 and I_5 invariants involve second and third derivatives; both are difficult to reconstruct numerically, and in this sense the evaluation of invariants is a difficult task for the simulation.

Table 1. Invariant values (absolute value).

Invar.	Analytical Value	$L_\infty^{t=0}$	$L_\infty^{t=20}$	$L_\infty^{t=40}$
I_1	7.200	$5.272 \cdot 10^{-5}$	$1.675 \cdot 10^{-5}$	$5.344 \cdot 10^{-5}$
I_2	2.592	$7.840 \cdot 10^{-11}$	$6.146 \cdot 10^{-7}$	$6.406 \cdot 10^{-7}$
I_3	4.666	$6.304 \cdot 10^{-3}$	$6.302 \cdot 10^{-3}$	$6.308 \cdot 10^{-3}$
I_4	23.131	$1.226 \cdot 10^{-5}$	$3.557 \cdot 10^{-5}$	$3.719 \cdot 10^{-4}$
I_5	102.792	$5.018 \cdot 10^{-4}$	$4.910 \cdot 10^{-4}$	$3.062 \cdot 10^{-3}$

EXAMPLE 1.2. INTERACTION BETWEEN SOLITONS. The interaction between two or more solitons is numerically important. Higher solitons move with greater velocity than lower ones, so when a higher soliton reaches a lower one, it overtakes it and remains unchanged after the interaction. This is not due to a linear superposition principle, as, after the overtaking, the two waves are phase-shifted and they are not in the position, after the interaction, which would be anticipated if each moved at a constant speed throughout the collision. In this example, we show a two-soliton interaction for $m = 2$. According to Djidjeli *et al.* [10], we choose

$$u(t,x) = \sum_{i=1}^{2} \left[A_i \operatorname{sech}^2(\kappa_i x - \omega_i t - x_i)\right]^{1/m} \tag{3.4}$$

as the initial condition, where $A_i = 2(m+1)(m+2)m^{-2}\mu\kappa_i^2$ and $\omega_i = 4\mu\kappa_i^3 m^{-2}$. Figure 2 shows the simulation carried out with $n = 80$ and $\Delta t = 0.01$, $x \in [-70, 70]$, $t \in [0, 360]$, $\mu = 1$, $\kappa_1 = 0.3$, $\kappa_2 = 0.2$, $x_1 = -2$, and $x_2 = 3$. The CPU time in this case is 42.5 s. The higher soliton reaches the smaller and slower one and, after interaction, it emerges with the same shape. In this case, the exact solution is not known, and therefore, we are not able to predict the exact value of the conserved densities; nevertheless, a numerical check shows that the invariant quantities I_1 and I_2 remain unchanged during the simulation within an interval of less than $\pm 10^{-3}$ with respect to the value of $t = 0$.

EXAMPLE 2. FIFTH-ORDER KORTEWEG-DE VRIES EQUATION. A natural extension of the third-order KdV equation is the fifth-order KdV equation.

$$u_t + uu_x - u_{xxxxx} = 0. \tag{3.5}$$

Numerical simulations of this model are very delicate due the presence of the fifth-order derivative and, at the same time, the equation presents some very interesting behavior peculiarities [25].

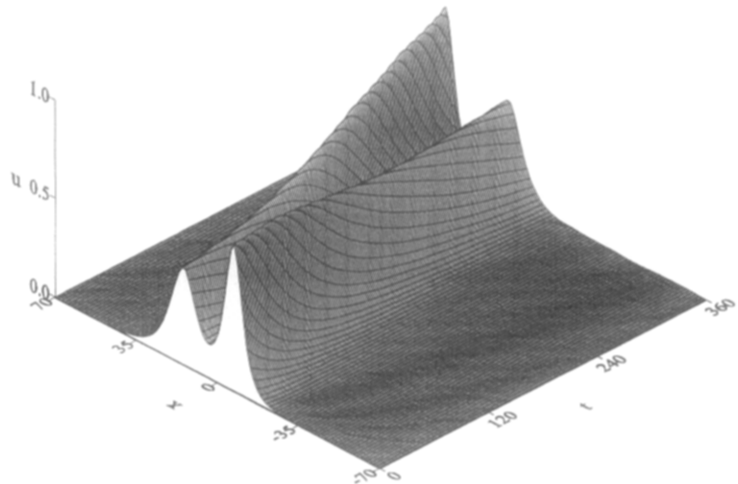

Figure 2. Generalized third-order KdV equation with $m = 2$: interaction between two solitons.

EXAMPLE 2.1. ONE SOLITON. The exact solution is also unknown in this case; the initial condition used by Nagashima and Kuwahara [25] is considered here,

$$\begin{aligned} u_{t=0}(x) = 2.65759\lambda e^{-0.16(x-x_0)^2\sqrt{\lambda}} \Big[& 1.00079 - 6.76159 \cdot 10^{-3}\lambda^{1/2}(x-x_0)^2 \\ & - 1.35573 \cdot 10^{-3}\lambda(x-x_0)^4 + 2.52023 \cdot 10^{-5}\lambda^{3/2}(x-x_0)^6 - 4.78259 \cdot 10^{-6}\lambda^2(x-x_0)^8 \Big]. \end{aligned} \tag{3.6}$$

Figure 3 shows the result obtained with $\lambda = 0.3$, $x_0 = 0$, $-35 < x < 35$, and $0 < t < 40$. The number n of nodes is 60 and $\Delta t = 0.005$. The CPU time in this case is 12.6 s. The fifth-order KdV equation only involves three invariant quantities, that is, the I_1, I_2, and $I_3' = \int_{-\infty}^{+\infty}((1/3)u^3 - u_{xx}^2)\,dx$. The values of these quantities obtained during the simulation are given in Table 2, and testify how they remain unchanged with a good precision.

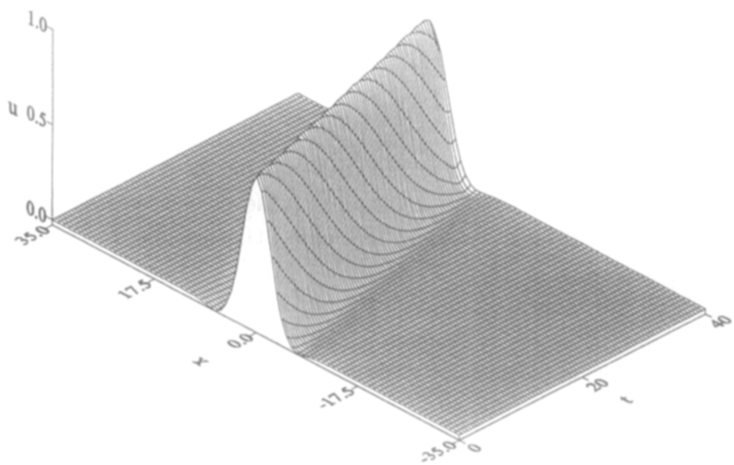

Figure 3. Fifth-order KdV equation with $m = 1$: one soliton.

EXAMPLE 2.2. INTERACTION BETWEEN SOLITONS. The interaction between two solitons is similar to that of the third-order KdV equation. We use the sum of two solitons such as (3.6) with $\lambda_1 = 0.3$, $\lambda_2 = 0.1$, $x_1 = -7$, and $x_2 = 12$ as the initial condition. The space and time domains are $[-20, 60]$ and $[0, 120]$, respectively. The number of nodes is $n = 60$ while $\Delta t = 0.005$. The CPU time is 18.0 s. The results are plotted in Figure 4 and, even in this case, the invariant quantities remain practically unchanged.

Table 2. Invariant values.

	$t = 0$	$t = 120$	$t = 240$	$t = 360$
I_1	4.31334	4.31429	4.31385	4.31225
I_2	1.29539	1.29539	1.29539	1.29539
I_3'	0.495342	0.495342	0.495342	0.495340

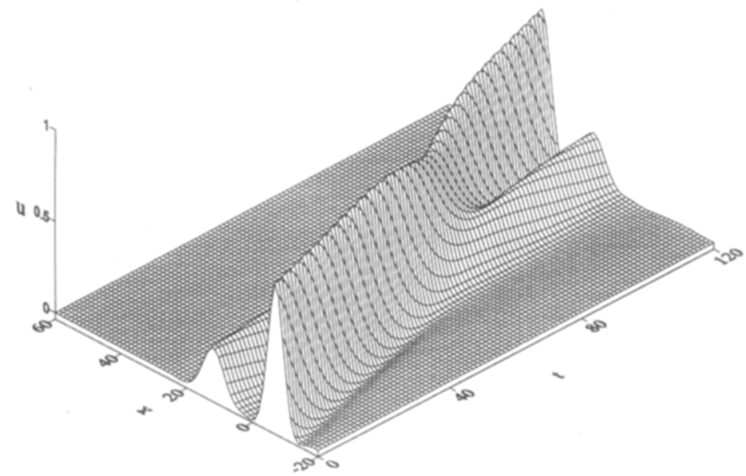

Figure 4. Fifth-order KdV equation with $m = 1$: interaction between two solitons.

EXAMPLE 3. VARIABLE COEFFICIENT KORTEWEG-DE VRIES EQUATION. The problem of the propagation of nonlinear dispersive waves over variable depths is a question that arises in many practical situations, and it has received a great deal of attention (e.g., [2]). In this case, the model is given by a variable-coefficient Korteweg-de Vries equation

$$2\sqrt{D}\eta_X + \frac{1}{2}\frac{D'}{\sqrt{D}}\eta + \frac{3}{D}\eta\eta_\xi + \frac{1}{3}D\eta_{\xi\xi\xi} = 0, \tag{3.7}$$

where $\xi = (1/\epsilon)\chi(X) - t$, $X = \epsilon x$, $\chi(X) = \int_0^X \frac{\mathrm{d}X'}{\sqrt{D(X')}}$, and $D(X)$ represent the local depth. It is known that the behavior of the solitary wave depends on the "velocity" of the change of the depth [2]. As an example, Figure 5 shows a soliton that moves over a "fast" depth variation from $D = 1$ $(X < 0)$ to $D_0 = 0.451$ $(X \geq 0)$. In this case, it is possible to demonstrate that three solitons should arise, according to $D_0 = [(1/2)N(N+1)]^{-4/9}]$ where N is the number of solitons that should arise after the depth variation. Paying attention to the scale change (according to (3.7)), we choose $\eta(\xi, X = 0) = A \cdot \mathrm{sech}^2((1/2)\sqrt{3A}\xi)$ as the initial boundary condition. Figure 5 shows the simulation carried out with $A = 1$, $n = 80$, and $\Delta t = 0.001$. The simulation domains are $0 < X < 2$ and $-5 < \xi < 10$. The CPU time is 3.5 s.

EXAMPLE 4. SINE-GORDON EQUATION. The sine-Gordon equation is a useful example to show how solitons arise with another type of nonlinearity [1,26]. The use of this equation also shows how the proposed method is able to simulate a second time derivative or a system of equations. The mathematical model

$$u_{tt} - u_{xx} + \sin u = 0 \tag{3.8}$$

can in fact be written as the system

$$\begin{aligned} u_t &= v, \\ v_t &= u_{xx} - \sin u. \end{aligned} \tag{3.9}$$

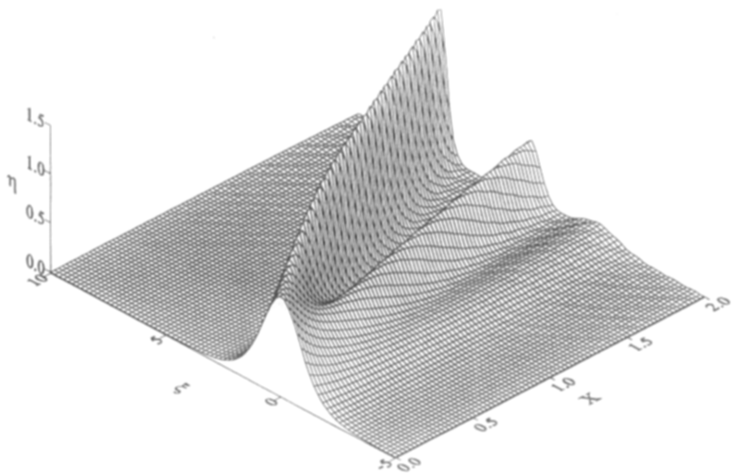

Figure 5. Variable coefficient KdV equation: soliton over a "fast" depth change.

There are some examples of the analytical solution of (3.8) in the literature [26]. Among these we choose the solution relative to the *kink-antikink* interaction. The analytical solution for this case is

$$u(x,t) = 4\arctan\left[c\sinh\left(\frac{1}{\sqrt{1-c^2}}x\right)\operatorname{sech}\left(\frac{c}{\sqrt{1-c^2}}t\right)\right]. \tag{3.10}$$

Figures 6 and 7 show the value of u and v, respectively. The parameters are $c = 0.5$, $x \in [-25, 25]$, and $t \in [-20, 20]$. The number of nodes is $n = 80$ and $\Delta t = 0.01$. The CPU time required is 4.7 s. The L_∞ for u and v are, respectively, 0.00982 and 0.0188.

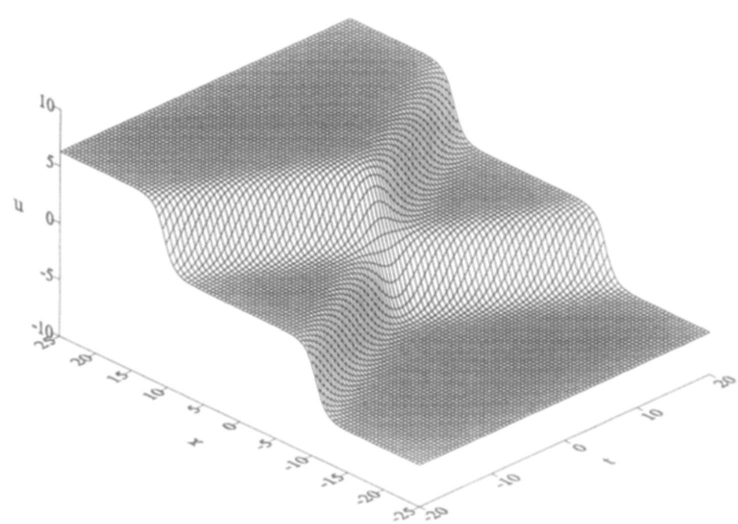

Figure 6. Sine-Gordon equation: kink-antikink interaction.

4. COMMENTS AND CONCLUSIONS

The examples reported in the previous section show how the generalized collocation-interpolation method based on the use of the Sinc function works well with such problems as the numerical simulation of the generalized KdV equation or the sine-Gordon equation. These are severe test cases; the nonlinearities and the presence of high order spatial derivatives or time derivatives make the numerical simulation of these models difficult.

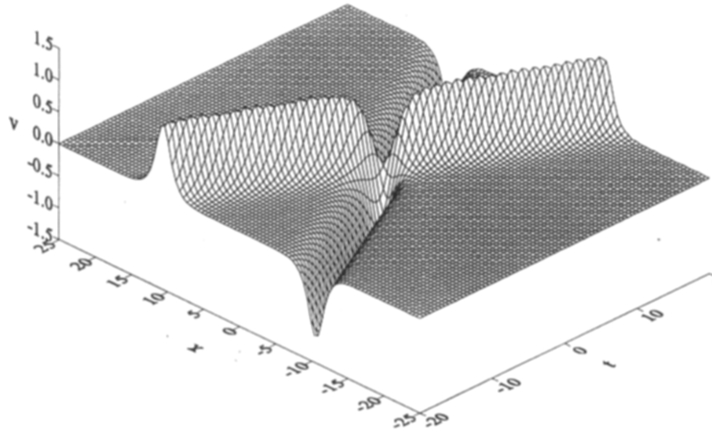

Figure 7. Sine-Gordon equation: kink-antikink interaction (time derivative).

The figures shown in the previous section testify to the good behavior of the solutions in the zones of the solution where there are both high and low gradients. The former occur, in particular, near the peak of the solitary waves; the latter occur near the boundary. The figures have been drawn up without any graphical interpolation and indicate good agreement near the peak and stable behavior of the solution in the flat zones. The errors are always very low, in spite of the fact that the examples are conducted—deliberately, to show the quality of the method—with a coarse grid. Furthermore, the tests on the high level invariants have also shown how the numerical calculation of the derivatives is precise and reliable. Due to the lack of space, only a few examples for each equation have been described here. However, extended numerical experimentation has shown that the method is particularly stable and the errors monotonically decrease with an increase in the number of nodes. The CPU time is always very low, and the proposed method is therefore very competitive compared to other methods, while the method itself is not limited, like other methods, by the choice of the number of nodes (like spectral methods that often require a number of nodes equal to 2^k). The proposed method can therefore be easily extended to other equations or systems of equations.

In conclusion, the generalized collocation-interpolation method based on the use of the Sinc function seems able to simulate nonlinear equations typical of the waves problems very well.

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